

Repeated multiplication by a constant ratio creates powerful patterns in sequences.

Geometric Sequences

From constant ratio to exponential behaviour.

Geometric sequences appear in nature, finance and many real-world models where change is multiplicative.

1. DEFINITION AND CONSTANT RATIO

A geometric sequence is a sequence where each term is obtained by multiplying the previous term by a constant ratio r .

$$u_{n+1} = ru_n$$

- r is the common ratio.
- Each term is a constant multiple of the previous term.
- The ratio between consecutive terms is the same.

$u_1, u_2, u_3, \dots$ is geometric if $\frac{u_{n+1}}{u_n} = r$ for all n .



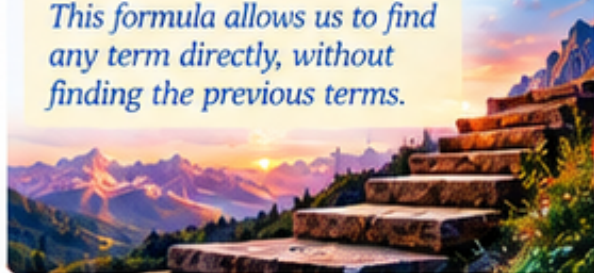
2. EXPLICIT FORMULA

The n^{th} term of a geometric sequence is given by

$$u_n = u_1 r^{n-1}$$

- u_1 is the first term.
- r is the common ratio (multiplied at each step).
- n is the term number.

This formula allows us to find any term directly, without finding the previous terms.



3. DETECTING GEOMETRIC STRUCTURE

A sequence is geometric if the ratio of consecutive terms is constant.

$$r = \frac{u_{n+1}}{u_n} \quad (u_n \neq 0)$$

Example: 3, 9, 27, 81, ...

$$\frac{9}{3} = 3, \quad \frac{27}{9} = 3, \quad \frac{81}{27} = 3$$

The ratio is constant ($r = 3$), so this is a geometric sequence.

Example: 2, 6, 12, 24, ...

$$\frac{6}{2} = 3, \quad \frac{12}{6} = 2 \quad (\text{ratio not constant})$$

so this is not a geometric sequence.

4. SPECIAL CASES

The value of r determines the behaviour of the sequence.

$r > 1$ Exponential growth. Terms increase without bound. (e.g. $r = 2$)

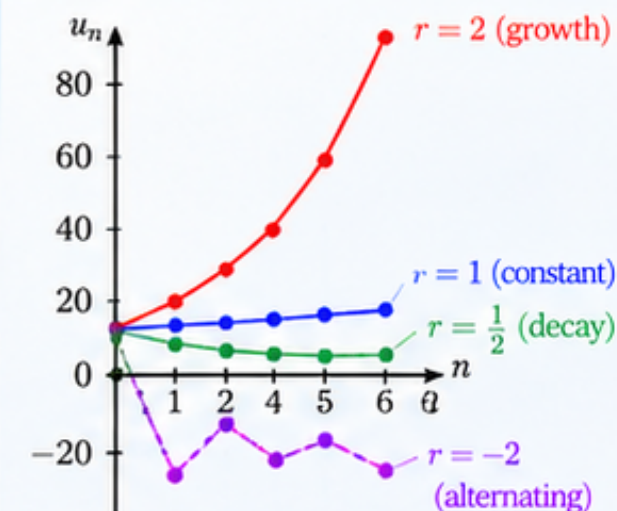
$0 < r < 1$ Exponential decay. Terms decrease towards 0. (e.g. $r = \frac{1}{2}$)

$r = 1$ Constant sequence. All terms are equal: $u_n = u_1$.

$r < 0$ Alternating signs. Terms change sign each step. (e.g. $r = -2$)

5. BEHAVIOUR AND VISUAL PATTERNS

Different values of r give different patterns of growth or decay.



6. FROM RECURSIVE TO EXPLICIT

Starting with $u_{n+1} = ru_n$, we can derive $u_n = u_1 r^{n-1}$.

$$\begin{aligned} u_2 &= ru_1 \\ u_3 &= ru_2 = r(ru_1) = r^2 u_1 \\ u_4 &= ru_3 = r^3 u_1 \\ &\vdots \\ u_n &= r^{n-1} u_1 \end{aligned}$$



Each step multiplies by r , so after $n - 1$ steps the term is $u_1 r^{n-1}$.

7. WORKED EXAMPLE: FIND TERMS

Find the first five terms of the geometric sequence with $u_1 = 4$ and $r = 3$.

Solution:

$$\begin{aligned} u_1 &= 4 \\ u_2 &= 3 \times 4 = 12 \\ u_3 &= 3 \times 12 = 36 \\ u_4 &= 3 \times 36 = 108 \\ u_5 &= 3 \times 108 = 324 \end{aligned}$$

The first five terms are 4, 12, 36, 108, 324.



8. FINDING A SPECIFIC TERM

Find u_{10} for the geometric sequence with $u_1 = 5$ and $r = 2$.

Solution:

$$\begin{aligned} u_n &= 5 \times 2^{n-1} \\ u_{10} &= 5 \times 2^9 \\ &= 5 \times 512 \\ &= 2560 \end{aligned}$$

So the 10th term is 2560.



9. FINDING THE RATIO OR FIRST TERM

A later term can be used to find the ratio r or the first term u_1 .

Example: Given $u_2 = 6$ and $u_3 = 18$, find r and u_1 .

Solution:

$$\begin{aligned} r &= \frac{u_3}{u_2} = \frac{18}{6} = 3. \\ u_1 &= \frac{u_2}{r} = \frac{6}{3} = 2. \end{aligned}$$

So $r = 3$ and $u_1 = 2$.



10. REAL-WORLD APPLICATIONS

Geometric sequences model situations where change is multiplicative.



Population growth
A population that multiplies by a constant factor each period.



Finance
Compound interest grows by a constant percentage each period.



Depreciation
The value of an asset decreases by a constant percentage each year.

11. ARITHMETIC VS GEOMETRIC CHANGE

Arithmetic sequences add a constant difference. Geometric sequences multiply by a constant ratio.

	Arithmetic	Geometric
Rule	$u_{n+1} = u_n + d$	$u_{n+1} = ru_n$
Change	Constant difference d	Constant ratio r
Explicit	$u_n = u_1 + (n - 1)d$	$u_n = u_1 r^{n-1}$
Example	2, 5, 8, 11, ... ($d = 3$)	2, 6, 18, 54, ... ($r = 3$)
Growth	Linear	Exponential

12. PAPER 3-STYLE CHALLENGE

A culture of bacteria starts with 500 cells and triples in number each hour.

- Write down a recurrence relation for the number of cells after n hours. [1]
- Find an explicit formula for u_n . [2]
- Determine the least value of n for which $u_n > 100\,000$. [2]
- Interpret your answer in the context of the model. [1]

Continue the full investigation in the accompanying teaching document.



13. BEYOND THE BASICS

- Connection to exponential functions
A geometric sequence is a discrete version of exponential growth: $u_n = u_1 r^{n-1}$.
- Using logarithms
Solve problems such as $u_n = k$ by using logarithms: $n = 1 + \frac{\log(k/u_1)}{\log r}$.
- Compound interest
 $A = P(1+i)^n$ follows a geometric model.
- Half-life and decay
Models of radioactive decay use $u_n = u_1 \left(\frac{1}{2}\right)^n$.
- Real-world modelling
Apply geometric sequences in finance, science and the environment.

14. LEARN MORE

- Student notes (full topic)
- Worked examples
- Guided practice worksheets
- Homework sets
- HL extension (deeper problems)
- Paper 3 investigation (full question)
- Solutions and markscheme (teacher version)
- Data sets and real-world examples
- Links to interactive tools (Desmos, GeoGebra)



Scan to open companion documents

Geometric Sequences

From constant ratio to exponential behaviour

IBAA-DIM-GS-001 – Lecture Note + Worksheet – Master Protocol v1.2

Learning goals

By the end of this unit, students should be able to recognise geometric structure from a constant ratio, move between recursive and explicit formulas, determine terms and parameters, interpret growth and decay, link geometric sequences to exponential functions, and model multiplicative change in context.

Two-Eyes check

Structure: Is each step obtained by multiplying by a constant factor?

Meaning: What do u_1 , r and n mean in the context?

Procedure: Do we need a recurrence, an explicit formula, or a ratio argument?

Optimisation: Can we check by generating a few terms, using a graph, or estimating size?

Learning route

We will move from recognising repeated multiplication to writing explicit formulas, then interpret geometric sequences as models for exponential growth and decay.

1. Guided lecture notes

A sequence (u_n) is **geometric** if the ratio of consecutive terms is constant:

$$\frac{u_{n+1}}{u_n} = r \quad (u_n \neq 0).$$

This gives the recurrence

$$u_{n+1} = ru_n.$$

Starting from the first term u_1 , repeated multiplication yields

$$u_2 = ru_1, \quad u_3 = r^2u_1, \quad \dots, \quad u_n = u_1r^{n-1}.$$

So the **explicit formula** is

$$u_n = u_1r^{n-1}.$$

Meaning before formula

A geometric sequence changes by the same *factor*, not the same amount. If bacteria triple each hour, the factor is 3. If a car keeps 85% of its value each year, the factor is 0.85.

Special cases:

- $r > 1$: exponential growth.
- $0 < r < 1$: exponential decay.
- $r = 1$: constant sequence.
- $r < 0$: terms alternate in sign.

Common errors

Do not confuse a common ratio with a common difference. Also, when the indexing starts at $n = 0$ rather than $n = 1$, adjust the exponent accordingly.

2. Worked examples

Example A. For 2, 6, 18, 54, . . . , $u_1 = 2$ and $r = 3$, so $u_n = 2 \cdot 3^{n-1}$.

Example B. If $u_1 = 5$ and $r = \frac{1}{2}$, then

$$u_6 = 5 \left(\frac{1}{2}\right)^5 = \frac{5}{32}.$$

Example C. If $u_2 = 6$ and $u_4 = 54$, then

$$\frac{u_4}{u_2} = r^2 = \frac{54}{6} = 9 \implies r = 3,$$

so $u_1 = 2$ and $u_n = 2 \cdot 3^{n-1}$.

Example D. If a population starts at 500 and triples every hour, then after n hours

$$P_n = 500 \cdot 3^n$$

when the initial amount is indexed by $n = 0$.

3. Core worksheet

1. Classify each sequence as geometric or not geometric. Give a reason. [6]

(a) 4, 12, 36, 108, . . . [2]

.....

(b) 7, 14, 28, 54, . . . [2]

.....

(c) 9, −18, 36, −72, . . . [2]

.....

2. For the geometric sequence with $u_1 = 3$ and $r = 2$, find: [6]

(a) the explicit formula. [2]

.....

(b) u_8 . [2]

.....

(c) u_{10} . [2]

.....

3. Given $u_2 = 10$ and $u_5 = 80$ for a geometric sequence, find r and u_1 . [5]

.....
.....
.....

4. A culture starts with 200 bacteria and doubles every 30 minutes. Let B_n be the population after n half-hours. [5]

(a) Write a recurrence and an explicit formula. [2]

.....

.....

(b) Find the population after 4 hours. [3]

.....

.....

5. Explain why the graph of a geometric sequence can suggest an exponential curve, even though the sequence itself is discrete. [3]

.....

.....

.....

4. HL extension / deeper thinking

Solving for the term number

If $u_n = u_1 r^{n-1}$ and we want to know when $u_n > k$, logarithms are often useful. For example, if $u_n = 4 \cdot 3^{n-1}$ and we seek the least n with $u_n > 1000$,

$$4 \cdot 3^{n-1} > 1000 \iff 3^{n-1} > 250 \iff n - 1 > \frac{\log 250}{\log 3}.$$

Hence the least integer n is 7.

5. Exit ticket and homework

- State the definition of a geometric sequence using a ratio.
- Write the explicit formula for a geometric sequence.
- Give one real-world example of multiplicative change.
- Homework: choose one growth and one decay example and model each with a geometric sequence.

6. IA / further investigation directions

Possible explorations

Explore compound interest, half-life, population models, repeated discounts, or branching patterns. A good investigation should compare model predictions with assumptions and limitations, not just compute terms.

7. Paper 3-style investigation

Poster Challenge ID: IBAA-DIM-GS-001-P3

A culture of bacteria starts with 500 cells and triples every hour. The poster asks for a recurrence, an explicit formula, a threshold question, and interpretation. This investigation develops that opening into a structured study of multiplicative growth.

Let u_n be the number of bacteria after n hours, with $u_0 = 500$.

1. Write a recurrence relation and an explicit formula for u_n . [4]

(a) Recurrence. [2]

.....

.....

(b) Explicit formula. [2]

.....

.....

2. Find the first six terms of the sequence and comment on the pattern of change. [4]

.....

.....

.....

.....

3. Determine the least value of n for which $u_n > 100000$. [5]

.....

.....

.....

.....

4. Suppose a nutrient limit means the observed count after 6 hours is only 300000 rather than the model prediction. [5]

(a) Find the model prediction at $n = 6$. [2]

.....

.....

(b) Explain what this discrepancy suggests about the model. [3]

.....

.....

.....

5. A second culture starts at 800 and doubles each hour. Let its size be v_n . [6]

(a) Write an explicit formula for v_n . [2]

.....

.....

(b) Compare u_4 and v_4 . [2]

.....
.....

(c) Explain structurally why a smaller starting value can still produce a larger value later. [2]

.....
.....
.....

6. Evaluate the strengths and limitations of geometric-sequence models for population growth. Give at least two strengths and two limitations, then propose one more realistic refinement. [6]

.....
.....
.....
.....
.....
.....